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Reformulated invariants for non-torus knots & links

Zodinmawia and P. Ramadevi

Department of Physics, Indian Institute of Technology Bombay,
Mumbai, India, 400076

zodin@phy.iitb.ac.in, ramadevi@phy.iitb.ac.in

Abstract: Using the Racah coefficients in our earlier paper arXiv:11073918, we explicitly write the Chern-Simons field theory invariants for many non-torus knot and links. Further, we have tabulated the reformulated invariants which agrees with the Ooguri-Vafa conjecture for knots and Labastida-Marino-Vafa conjecture for links.

Keywords: Chern-Simons field theory, Knot polynomials, Ooguri-Vafa conjecture.

Contents

1. Introduction

Starting with the pioneering work of Witten [1], there have interesting developments between Chern-Simons theory and knot theory. $U(N)$ Chern-Simons theory provides a pool of colored framed knot invariants $V_R[\mathcal{K}; p]$ given by the expectation value of Wilson loop operator $\mathcal{K}$, with framing number p, carrying representation R [2]. We can place different representations on the components of a r-component link giving multicolored framed link invariants $V_{R_1,R_2,\ldots R_r}[\mathcal{L}; p_1, p_2, \ldots p_r]$ where the r-tuple vector $\vec{p}$ denotes the framing numbers on the respective component knots. In this paper, we will write the invariants for zero framed knots (p=0) and zero framed 2-component links ($p_1 = p_2 = 0$).

The polynomial form of these colored invariants can be obtained for $(2, 2p + 1)$ class of torus knots and $(2, 2p)$ class of torus links [3, 4, 5, 6, 7]. For the most general class of (n, m) torus knots , the explicit polynomial form for fundamental representation ($R = \Box$) placed on the knot is given by Theorem 9.7 in Ref.[8] and from Chern-Simons theory [9]. Further, the colored invariants of (n, m) torus knots and links have been discussed in Refs.[10, 11, 12, 13, 14].

For non-torus knots and non-torus links, we could formally write an expression for these invariants which involves quantum Racah coefficients [3, 4]. Plat diagrams for various non-torus knots and links are given in Figure 1 and Figure 2. Recently, the colored invariants of figure eight ($\mathbf{4_1}$) knot carrying totally symmetric or totally antisymmetric representation has been conjectured [15]. However, for other non-torus knots we could

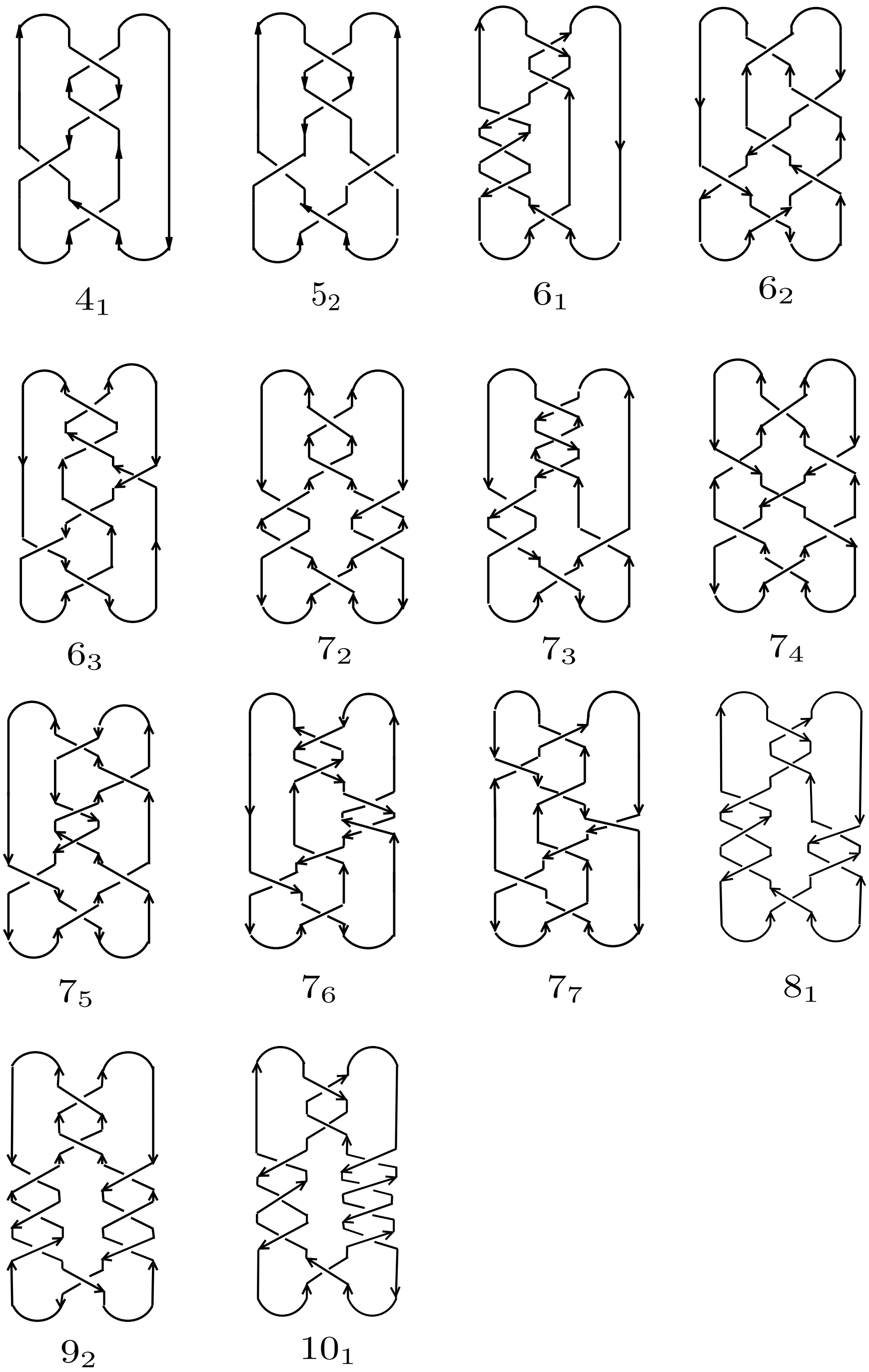

Figure 1: Non-torus knots.

write the explicit polynomial form only for some representations using the Racah coefficients [16]. We present, for the knots and links in Figures 1, 2, the polynomial form of the colored knot invariants and multicolored two-component link invariants.

Using these colored knot and multicolored link invariants, we tabulate the reformulated invariants. The polynomial form of these reformulated invariants indeed agrees with the

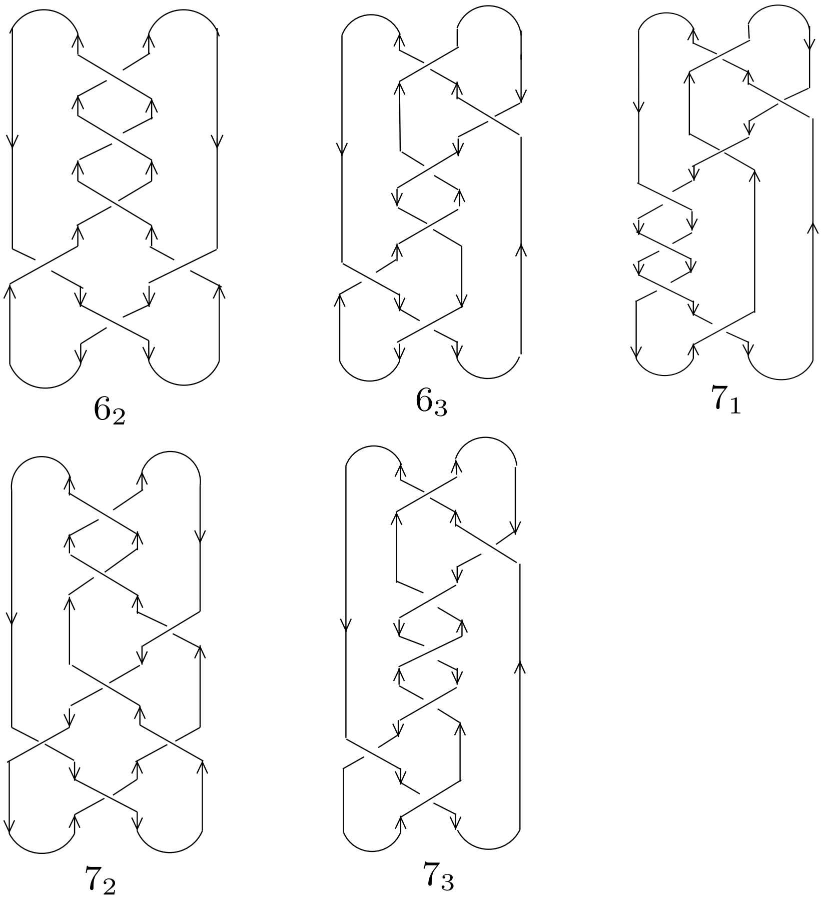

Figure 2: Non-torus links.

Ooguri-Vafa conjecture for knots [17] and Labastida-Marino-Vafa for links [18]. The results indirectly confirms that the Racah coefficients obtained in [16] from the equivalence of knots or links are correct.

The plan of the paper is as follows: In section 2, we present the Chern-Simons invariants of the non-torus knots and links in Figures 1, 2. Then in section 3, we write down the polynomial form of these invariants for some representations. Using the polynomial form, we obtain the reformulated polynomials in section 4. We find the reformulated polynomials of knots obey Ooguri-Vafa conjecture and that of links obey Labastida-Marino-Vafa conjecture. Finally, we present some of the challenging open problems in the concluding section.

2. U(N) invariants in terms of the quantum Racah coefficients

$U(N)$ Chern-Simons invariant, for any knot or link, is a product of $SU(N)$ and the $U(1)$ invariant [2]. The $U(1)$ link invariant is related to the linking and self linking number of the component knots. With a suitable choice of $U(1)$ charge [19], the $U(N)$ invariants are polynomials in variable $q = e^{\frac{2\pi i}{k+N}}$ and $\lambda = q^N$ where k is the Chern-Simons coupling.

Following Ref.[3], the $SU(N)$ link invariant can be directly written down using the two inputs:(1) The relation between $SU(N)$ Chern Simons field theory on a three, ball with $SU(N)_k$ conformal field theory on the boundary of the three-ball [1], (2) any knot or and link can be obtained as platting of braids (Birman theorem) [20]. Explicit discussion of this method can be found in [16] where we have obtained the $U(N)$ invariant for the framed knot 5_2 and the framed link 6_2. Particularly, the braiding eigenvalues of parallely oriented strands $\lambda_s^{(+)}(R_1, R_2)$ and antiparallely oriented strands $\lambda_s^{(-)}(R_1, \bar{R}_2)$ (see equation (2.14) in [16]) and the Racah coefficients presented [16] will be useful to write the polynomial form for some representations.

For completeness, we present the expression of $U(N)$ link invariant for non torus knots and links in Figure 1 and Figure 2 in terms of $SU(N)$ quantum Racah coefficients and braiding eigenvalues.

2.1 Non Torus Knots

In these formulae, l denotes the total number of boxes in the Young Tableaux representation R and $\kappa_R = \frac{1}{2}[Nl + l + \sum_i (l_i^2 - 2il_i)]$ where l_i is the number of boxes in the i-th row. Using the braiding eigenvalues, Racah coefficients, κ_R and l, we write the $U(N)$ Chern-Simons invariant for zero-framed non-torus knots:

$$V_R^{\{U(N)\}}[4_1;\, 0] = \sum_{s,t,s'} \epsilon_s^{\bar{R},R} \sqrt{dim_q s}\, \epsilon_{s'}^{R,R} \sqrt{dim_q s'}\, (\lambda_s^{(-)}(\bar{R},\, R))^2\, a_{ts} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix}$$
$$(\lambda_t^{(-)}(\bar{R},\, R))^{-1}\, a_{ts'} \begin{bmatrix} \bar{R} & R \\ R & \bar{R} \end{bmatrix} (\lambda_{s'}^{(+)}(R,\, R))^{-1}. \tag{2.1}$$

$$V_R^{\{U(N)\}}[5_2;\, 0] = q^{(-5\kappa_R + \frac{5l^2}{2N})} \sum_{s,t,s'} \epsilon_s^{R,R} \sqrt{dim_q s}\, \epsilon_{s'}^{R,R} \sqrt{dim_q s'}\, (\lambda_s^{(+)}(R,\, R))^{-2}\, a_{ts} \begin{bmatrix} \bar{R} & R \\ R & \bar{R} \end{bmatrix}$$
$$(\lambda_t^{(-)}(\bar{R},\, R))^{-2}\, a_{ts'} \begin{bmatrix} R & \bar{R} \\ \bar{R} & R \end{bmatrix} (\lambda_{s'}^{(+)}(R,\, R))^{-1}. \tag{2.2}$$

$$V_R^{\{U(N)\}}[6_1;\, 0] = q^{(-2\kappa_R + \frac{l^2}{N})} \sum_{s,t,s'} \epsilon_s^{\bar{R},R} \sqrt{dim_q s}\, \epsilon_{s'}^{R,R} \sqrt{dim_q s'}\, (\lambda_s^{(-)}(\bar{R},\, R))^2\, a_{ts} \begin{bmatrix} R & \bar{R} \\ R & \bar{R} \end{bmatrix}$$
$$(\lambda_t^{(-)}(\bar{R},\, R))^{-3}\, a_{ts'} \begin{bmatrix} \bar{R} & R \\ R & \bar{R} \end{bmatrix} (\lambda_{s'}^{(+)}(R,\, R))^{-1}. \tag{2.3}$$

$$V_R^{\{U(N)\}}[6_2;\, 0] = q^{(-2\kappa_R + \frac{l^2}{N})} \sum_{s,t,s',u,v} \epsilon_s^{R,R} \sqrt{dim_q s}\, \epsilon_v^{\bar{R},R} \sqrt{dim_q v}\, \lambda_s^{(+)}(R,\, R)\, a_{ts} \begin{bmatrix} \bar{R} & R \\ R & \bar{R} \end{bmatrix}$$
$$\lambda_t^{(-)}(R,\, \bar{R})\, a_{ts'} \begin{bmatrix} \bar{R} & R \\ \bar{R} & R \end{bmatrix} (\lambda_{s'}^{(-)}(R,\, \bar{R}))^{-1}\, a_{us'} \begin{bmatrix} \bar{R} & \bar{R} \\ R & R \end{bmatrix}$$
$$(\lambda_u^{(+)}(R,\, R))^{-2} a_{uv} \begin{bmatrix} \bar{R} & \bar{R} \\ R & R \end{bmatrix} (\lambda_v^{(-)}(\bar{R},\, R))^{-1}. \tag{2.4}$$

$$V_R^{\{U(N)\}}[6_3;\,0] = \sum_{s,t,s',u,v} \epsilon_s^{R,R}\,\sqrt{dim_q s}\,\epsilon_v^{\bar{R},R}\,\sqrt{dim_q v}\,(\lambda_s^{(+)}(R,\,R))^{-2}\,a_{ts}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}$$
$$(\lambda_t^{(-)}(R,\,\bar{R}))^{-1}\,a_{ts'}\begin{bmatrix}\bar{R} & R\\ \bar{R} & R\end{bmatrix}\,\lambda_{s'}^{(-)}(R,\,\bar{R})\,a_{us'}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}$$
$$(\lambda_u^{(+)}(\bar{R},\,\bar{R}))a_{uv}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}\,(\lambda_v^{(-)}(\bar{R},\,R)). \tag{2.5}$$

$$V_R^{\{U(N)\}}[7_2;\,0] = q^{(-7\kappa_R+\frac{7l^2}{2N})}\sum_{s,t,s'} \epsilon_s^{R,R}\,\sqrt{dim_q s}\,\epsilon_{s'}^{R,R}\,\sqrt{dim_q s'}\,(\lambda_s^{(+)}(R,\,R))^{-2}\,a_{ts}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}$$
$$(\lambda_t^{(-)}(R,\,\bar{R}))^{-4}\,a_{ts'}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}\,(\lambda_{s'}^{(+)}(R,\,R))^{-1}. \tag{2.6}$$

$$V_R^{\{U(N)\}}[7_3;\,0] = q^{(7\kappa_R-\frac{7l^2}{2N})}\sum_{s,t,s'} \epsilon_s^{R,\bar{R}}\,\sqrt{dim_q s}\,\epsilon_{s'}^{R,\bar{R}}\,\sqrt{dim_q s'}\,(\lambda_s^{(-)}(R,\,\bar{R}))^{3}\,a_{ts}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}$$
$$(\lambda_t^{(+)}(\bar{R},\,\bar{R}))^{3}\,a_{ts'}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}\,\lambda_{s'}^{(-)}(R,\,\bar{R}). \tag{2.7}$$

$$V_R^{\{U(N)\}}[7_4,;\,0] = q^{(7\kappa_R-\frac{7l^2}{2N})}\sum_{s,t,s',u,v} \epsilon_s^{R,R}\,\sqrt{dim_q s}\,\epsilon_v^{R,R}\,\sqrt{dim_q v}\,\lambda_s^{(+)}(R,\,R)\,a_{ts}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}$$
$$(\lambda_t^{(-)}(R,\,\bar{R}))^{2}\,a_{ts'}\begin{bmatrix}R & \bar{R}\\ \bar{R} & R\end{bmatrix}\,\lambda_{s'}^{(+)}(\bar{R},\,\bar{R})\,a_{us'}\begin{bmatrix}R & \bar{R}\\ \bar{R} & R\end{bmatrix}$$
$$(\lambda_u^{(-)}(R,\,\bar{R}))^{2}\,a_{uv}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}\,\lambda_v^{(+)}(R,\,R). \tag{2.8}$$

$$V_R^{\{U(N)\}}[7_5;\,0] = q^{(-7\kappa_R+\frac{7l^2}{2N})}\sum_{s,t,s',u,v} \epsilon_s^{R,\bar{R}}\,\sqrt{dim_q s}\,\epsilon_v^{R,\bar{R}}\,\sqrt{dim_q v}\,(\lambda_s^{(-)}(R,\,R))^{-1}\,a_{ts}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}$$
$$(\lambda_t^{(+)}(R,\,R))^{-1}\,a_{ts'}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}\,(\lambda_{s'}^{(-)}(\bar{R},\,R))^{-2}\,a_{us'}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}$$
$$\lambda_u^{(+)}(R,\,R)\,a_{uv}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}\,(\lambda_v^{(-)}(\bar{R},\,R))^{-1}. \tag{2.9}$$

$$V_R^{\{U(N)\}}[7_6;\,0] = q^{(-3\kappa_R+\frac{3l^2}{2N})}\sum_{s,t,s',u,v} \epsilon_s^{R,\bar{R}}\,\sqrt{dim_q s}\,\epsilon_v^{\bar{R},R}\,\sqrt{dim_q v}\,(\lambda_s^{(-)}(R,\,\bar{R}))^{-2}\,a_{ts}\begin{bmatrix}\bar{R} & R\\ \bar{R} & R\end{bmatrix}$$
$$(\lambda_t^{(-)}(\bar{R},\,R))^{2}\,a_{ts'}\begin{bmatrix}\bar{R} & R\\ \bar{R} & R\end{bmatrix}\,(\lambda_{s'}^{(-)}(R,\,\bar{R}))^{-1}\,a_{us'}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}$$
$$(\lambda_u^{(+)}(\bar{R},\,\bar{R}))^{-1}\,a_{uv}\begin{bmatrix}\bar{R} & \bar{R}\\ R & R\end{bmatrix}\,(\lambda_v^{(-)}(\bar{R},\,R))^{-1}. \tag{2.10}$$

$$
\begin{aligned}
V_R^{\{U(N)\}}[7_7;\,0] =& q^{(\kappa_R-\frac{l^2}{2N})} \sum_{s,t,s',u,v,w,x} \epsilon_s^{R,R}\sqrt{dim_q s}\,\epsilon_v^{R,R}\sqrt{dim_q v}\,(\lambda_s^{(+)}(R,\,R))\,a_{ts}\begin{bmatrix}\bar{R} & R\\ \bar{R} & R\end{bmatrix}\\
&(\lambda_t^{(-)}(\bar{R},\,R))\,a_{ts'}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}(\lambda_{s'}^{(-)}(\bar{R},\,R))^{-1}\,a_{us'}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix}\\
&(\lambda_u^{(+)}(\bar{R},\,\bar{R}))^{-1}\,a_{uv}\begin{bmatrix}R & R\\ \bar{R} & \bar{R}\end{bmatrix}(\lambda_v^{(-)}(R,\,\bar{R}))^{-1}a_{wv}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix}\\
&\lambda_w^{(-)}(R,\bar{R})\,a_{wx}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}\lambda_x^{(+)}(R,R).
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
V_R^{\{U(N)\}}[8_1;\,0] =& q^{(-4\kappa_R+\frac{2l^2}{N})} \sum_{s,t,s'} \epsilon_s^{\bar{R},R}\sqrt{dim_q s}\,\epsilon_{s'}^{R,R}\sqrt{dim_q s'}\,(\lambda_s^{(-)}(\bar{R},\,R))^2\,a_{ts}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix}\\
&(\lambda_t^{(-)}(\bar{R},\,R))^{-5}\,a_{ts'}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}(\lambda_{s'}^{(+)}(R,\,R))^{-1}.
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
V_R^{\{U(N)\}}[9_2;\,0] =& q^{(-7\kappa_R+\frac{7l^2}{2N})} \sum_{s,t,s'} \epsilon_s^{R,R}\sqrt{dim_q s}\,\epsilon_{s'}^{R,R}\sqrt{dim_q s'}\,(\lambda_s^{(+)}(R,\,R))^{-2}\,a_{ts}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}\\
&(\lambda_t^{(-)}(\bar{R},\,R))^{-6}\,a_{ts'}\begin{bmatrix}R & \bar{R}\\ \bar{R} & R\end{bmatrix}(\lambda_{s'}^{(+)}(R,\,R))^{-1}.
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
V_R^{\{U(N)\}}[10_1;\,0] =& q^{(-6\kappa_R+\frac{3l^2}{N})} \sum_{s,t,s'} \epsilon_s^{\bar{R},R}\sqrt{dim_q s}\,\epsilon_{s'}^{R,R}\sqrt{dim_q s'}\,(\lambda_s^{(-)}(\bar{R},\,R))^2\,a_{ts}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix}\\
&(\lambda_t^{(-)}(\bar{R},\,R))^{-7}\,a_{ts'}\begin{bmatrix}\bar{R} & R\\ R & \bar{R}\end{bmatrix}(\lambda_{s'}^{(+)}(R,\,R))^{-1}.
\end{aligned}
\tag{2.14}
$$

For framed knots $\mathcal{K}$ with framing number p, the invariants will be related to the zero-framed knot invariants as

$$
V_R^{\{U(N)\}}[\mathcal{K};\,p] = q^{p\kappa_R}V_R^{\{U(N)\}}[\mathcal{K};\,0]\ . \tag{2.15}
$$

From the formal expression for these invariants, we see that they involve two types of quantum $SU(N)$ Racah coefficients: (1) $a_{ts}\begin{bmatrix}R & \bar{R}\\ R & \bar{R}\end{bmatrix}$ which is a symmetric matrix and (2) $a_{uv}\begin{bmatrix}R & R\\ \bar{R} & \bar{R}\end{bmatrix}$ which are known only for $R = \Box, \Box\Box$ and $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$[16]. It is a challenging problem to find the Racah coefficients for other representations.

2.2 Non Torus Links

In the context of links, we can place different representations on the component knots. The invariants are hence called multicolored links. We present the $U(N)$ Chern-Simons

invariants for the two-component non-torus links(with framing zero) in Figure 2.

$$V^{\{U(N)\}}_{(R_1,R_2)}[6_2;\,0,0] = q^{\frac{3l_{R_1}l_{R_2}}{N}} \sum_{s,t,s'} \epsilon_s^{R_1,R_2}\,\sqrt{dim_q s}\,\epsilon_{s'}^{\bar{R}_1,\bar{R}_2}\,\sqrt{dim_q s'}\,(\lambda_s^{(+)}(R_1,\,R_2))^{-3}\,a_{ts}\begin{bmatrix} R_2 & \bar{R}_1 \\ \bar{R}_2 & R_1 \end{bmatrix}$$
$$(\lambda_t^{(-)}(\bar{R}_1,\,R_2))^{-2}\,a_{ts'}\begin{bmatrix} \bar{R}_1 & R_2 \\ R_1 & \bar{R}_2 \end{bmatrix}\,(\lambda_{s'}^{(+)}(\bar{R}_1,\,\bar{R}_2))^{-1}. \tag{2.16}$$

$$V^{\{U(N)\}}_{(R_1,R_2)}[6_3;\,0,0] = q^{(-2\kappa_{R_2}+\frac{l_{R_2}^2}{N}-\frac{2l_{R_1}l_{R_2}}{N})} \sum_{s,t,s',u,v} \epsilon_s^{R_1,R_2}\,\sqrt{dim_q s}\,\epsilon_v^{\bar{R}_1,\bar{R}_2}\,\sqrt{dim_q v}\,\lambda_s^{(+)}(R_1,\,R_2)$$
$$a_{ts}\begin{bmatrix} \bar{R}_1 & R_2 \\ R_1 & \bar{R}_2 \end{bmatrix}\,\lambda_t^{(-)}(R_1,\,\bar{R}_2)\,a_{ts'}\begin{bmatrix} \bar{R}_1 & R_2 \\ \bar{R}_2 & R_1 \end{bmatrix}\,(\lambda_{s'}^{(-)}(R_2,\,\bar{R}_2))^{-2}$$
$$a_{us'}\begin{bmatrix} \bar{R}_1 & R_2 \\ \bar{R}_2 & R_1 \end{bmatrix}\,\lambda_u^{(-)}(\bar{R}_1,\,R_2)\,a_{uv}\begin{bmatrix} R_2 & \bar{R}_1 \\ \bar{R}_2 & R_1 \end{bmatrix}\,\lambda_v^{(+)}(\bar{R}_1,\,\bar{R}_2). \tag{2.17}$$

$$V^{\{U(N)\}}_{(R_1,R_2)}[7_1;\,0,0] = q^{(-\kappa_{R_2}+\frac{l_{R_2}^2}{2N}+\frac{l_{R_1}l_{R_2}}{N})} \sum_{s,t,s',u,v} \epsilon_s^{R_1,R_2}\,\sqrt{dim_q s}\,\epsilon_v^{\bar{R}_1,R_2}\,\sqrt{dim_q v}\,(\lambda_s^{(+)}(R_1,\,R_2))$$
$$a_{ts}\begin{bmatrix} \bar{R}_1 & R_2 \\ R_1 & \bar{R}_2 \end{bmatrix}\,\lambda_t^{(-)}(R_1,\,\bar{R}_2)\,a_{ts'}\begin{bmatrix} \bar{R}_1 & R_2 \\ \bar{R}_2 & R_1 \end{bmatrix}\,(\lambda_{s'}^{(-)}(R_2,\,\bar{R}_2))^{-1}$$
$$a_{us'}\begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ R_2 & R_1 \end{bmatrix}\,\lambda_u^{(+)}(\bar{R}_1,\,\bar{R}_2)^{-3}\,a_{uv}\begin{bmatrix} \bar{R}_2 & \bar{R}_1 \\ R_2 & R_1 \end{bmatrix}\,(\lambda_v^{(-)}(\bar{R}_1,\,R_2))^{-1}. \tag{2.18}$$

$$V^{\{U(N)\}}_{(R_1,R_2)}[7_2;\,0,0] = q^{(-\kappa_{R_2}+\frac{l_{R_2}^2}{2N}-\frac{l_{R_1}l_{R_2}}{N})} \sum_{s,t,s',u,v} \epsilon_s^{R_1,R_2}\,\sqrt{dim_q s}\,\epsilon_v^{\bar{R}_1,R_2}\,\sqrt{dim_q v}\,(\lambda_s^{(+)}(R_1,\,R_2))^{-2}$$
$$a_{ts}\begin{bmatrix} \bar{R}_1 & R_1 \\ R_2 & \bar{R}_2 \end{bmatrix}\,(\lambda_t^{(-)}(R_2,\,\bar{R}_2))^{-1}\,a_{ts'}\begin{bmatrix} \bar{R}_1 & R_1 \\ \bar{R}_2 & R_2 \end{bmatrix}\,(\lambda_{s'}^{(-)}(R_1,\,\bar{R}_2))$$
$$a_{us'}\begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ R_1 & R_2 \end{bmatrix}\,\lambda_{\bar{u}}^{(+)}(R_1,\,R_2)^2\,a_{uv}\begin{bmatrix} \bar{R}_2 & \bar{R}_1 \\ R_2 & R_1 \end{bmatrix}\,\lambda_v^{(-)}(\bar{R}_1,\,R_2). \tag{2.19}$$

$$V^{\{U(N)\}}_{(R_1,R_2)}[7_3;\,0,0] = q^{(-3\kappa_{R_2}+\frac{3l_{R_2}^2}{2N})} \sum_{s,t,s',u,v} \epsilon_s^{R_1,R_2}\,\sqrt{dim_q s}\,\epsilon_v^{\bar{R}_1,R_2}\,\sqrt{dim_q v}\,\lambda_s^{(+)}(R_1,\,R_2)$$
$$a_{ts}\begin{bmatrix} \bar{R}_1 & R_2 \\ R_1 & \bar{R}_2 \end{bmatrix}\,\lambda_t^{(-)}(R_1,\,\bar{R}_2)\,a_{ts'}\begin{bmatrix} \bar{R}_1 & R_2 \\ \bar{R}_2 & R_1 \end{bmatrix}\,(\lambda_{s'}^{(-)}(R_2,\,\bar{R}_2))^{-3}$$
$$a_{us'}\begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ R_2 & R_1 \end{bmatrix}\,(\lambda_u^{(+)}(\bar{R}_1,\,\bar{R}_2))^{-1}\,a_{uv}\begin{bmatrix} \bar{R}_2 & \bar{R}_1 \\ R_2 & R_1 \end{bmatrix}$$
$$(\lambda_v^{(-)}(\bar{R}_1,\,R_2))^{-1}. \tag{2.20}$$

Including the framing numbers p_1, p_2 on the component knots of these two-component torus links $\mathcal{L}$, the framed multicolored invariant will be

$$V_{(R_1,R_2)}^{\{U(N)\}}[\mathcal{L};\, p_1, p_2] = q^{(p_1\kappa_{R_1}+p_2\kappa_{R_2})}\; V_{(R_1,R_2)}^{\{U(N)\}}[\mathcal{L};\, 0, 0]\ . \tag{2.21}$$

Even though the formula denoting the invariants of all these non-torus knots and links are available, we cannot write the explicit polynomial form for any representation placed on component knots because Racah coefficients is known only for certain representations. So, in the following section, we shall write the polynomial form of these invariants for those representations whose Racah coefficients are known [16].

3. Knot Polynomials

In this section we present the polynomial form of the $U(N)$ link invariant for non torus knots in Figure 1 for representation whose Young tableau diagrams are $\square$ and $\square\square$. The polynomial corresponding to representation $\square$ is proportional to HOMFLY-PT polynomial $P(\lambda, t)[K]$ [21, 22] upto unknot U normalisation:

$$P(\lambda, q)[K] = \frac{V_{\square}^{U(N)}[K;0]}{V_{\square}^{U(N)}[U]} = \frac{(q^{1/2}-q^{-1/2})}{(\lambda^{1/2}-\lambda^{-1/2})} V_{\square}^{U(N)}[K;0]\ . \tag{3.1}$$

We list them so that we can directly use them in the computation of reformulated invariants in section 5.

1. For fundamental representation $R = \square$ placed on the knot, the $U(N)$ knot polynomials are

$$V_{\square}^{U(N)}[4_1] = \frac{(\lambda-1)}{\lambda^{3/2}(q-1)\sqrt{q}}\left[-\lambda-\lambda q^2+\left(\lambda^2+\lambda+1\right)q\right]$$

$$V_{\square}^{U(N)}[5_2] = \frac{1}{(-1+q)\sqrt{q}\lambda^{7/2}}\left[q-q\lambda^3+\lambda(-1+\lambda^2)+q^2\lambda(-1+\lambda^2)\right]$$

$$V_{\square}^{U(N)}[6_1] = \frac{1}{\lambda^{5/2}(q-1)\sqrt{q}}\left[-\lambda^3+\lambda+(\lambda-\lambda^3)q^2+(\lambda^4+\lambda^3-\lambda^2-1)q\right]$$

$$V_{\square}^{U(N)}[6_2] = \frac{(-1+\lambda)}{(-1+q)q^{3/2}\lambda^{5/2}}\left[-\lambda-q^4\lambda-q^2(1+2\lambda)+q(1+\lambda+\lambda^2)+q^3(1+\lambda+\lambda^2)\right]$$

$$V_{\square}^{U(N)}[6_3] = -\frac{(-1+\lambda)}{(-1+q)q^{3/2}\lambda^{3/2}}\left[-\lambda-q^4\lambda+q(1+\lambda+\lambda^2)+q^3(1+\lambda+\lambda^2)-q^2(1+3\lambda+\lambda^2)\right]$$

$$V_{\square}^{U(N)}[7_2] = \frac{1}{(-1+q)\sqrt{q}\lambda^{9/2}}\left[\lambda(-1+\lambda^3)+q^2\lambda(-1+\lambda^3)-q(-1-\lambda^2+\lambda^3+\lambda^4)\right]$$

$$V_{\Box}^{U(N)}[7_3] = \frac{\lambda^{3/2}}{(-1+q)q^{3/2}}\big[-1+q+\lambda^2-q\lambda^3+q^4(-1+\lambda^2)+q^2(-1-\lambda+2\lambda^2)-q^3(-1+\lambda^3)\big]$$

$$V_{\Box}^{U(N)}[7_4] = \frac{(-1+\lambda)\lambda^{1/2}}{(-1+q)q^{1/2}}\big[(1+\lambda)^2+q^2(1+\lambda)^2-q(2+2\lambda+2\lambda^2+\lambda^3)\big]$$

$$V_{\Box}^{U(N)}[7_5] = \frac{(-1+\lambda)}{(-1+q)q^{3/2}\lambda^{9/2}}\big[\lambda(1+\lambda)+q^4\lambda(1+\lambda)-q(1+\lambda)^2-q^3(1+\lambda)^2+q^2(1+2\lambda+2\lambda^2)\big]$$

$$V_{\Box}^{U(N)}[7_6] = -\frac{(-1+\lambda)}{(-1+q)q^{3/2}\lambda^{7/2}}\big[\lambda^2+q^4\lambda^2-q\lambda(2+2\lambda+\lambda^2)-q^3\lambda(2+2\lambda+\lambda^2)+q^2(1+2\lambda+3\lambda^2+\lambda^3)\big]$$

$$V_{\Box}^{U(N)}[7_7] = \frac{(-1+\lambda)}{(-1+q)q^{3/2}\lambda^{3/2}}\big[\lambda+q^4\lambda-q(1+2\lambda+2\lambda^2)-q^3(1+2\lambda+2\lambda^2)+q^2(2+4\lambda+2\lambda^2+\lambda^3)\big]$$

$$V_{\Box}^{U(N)}[8_1] = \frac{1}{(q-1)\sqrt{q}\lambda^{7/2}}\big[-\lambda^4+\lambda+q^2(\lambda-\lambda^4)+q(\lambda^5+\lambda^4-\lambda^2-1)\big]$$

$$V_{\Box}^{U(N)}[9_1] = \frac{1}{(q-1)\sqrt{q}\lambda^{11/2}}\big[\lambda(\lambda^4-1)q^2-(\lambda^5+\lambda^4-\lambda^2-1)q+\lambda(\lambda^4-1)\big]$$

$$V_{\Box}^{U(N)}[10_1] = \frac{1}{(q-1)\sqrt{q}\lambda^{9/2}}\big[-\lambda^5+\lambda+q^2(\lambda-\lambda^5)+q(\lambda^6+\lambda^5-\lambda^2-1)\big]$$

2. For symmetric second rank representation $R = \Box\Box$, the knot polynomials are

$$V_{\Box\Box}^{U(N)}[4_1] = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^2q^2(1+q)\lambda^3}\big[(-1+\lambda)\lambda+3q^3\lambda^2-q^6(-1+\lambda)\lambda^2+q^4\lambda(-1+\lambda^2)+q^5\lambda^2(-1-\lambda+\lambda^2)-q(-1+\lambda+\lambda^2)+q^2(\lambda-\lambda^3)\big]$$

$$V_{\Box\Box}^{U(N)}[5_2] = \frac{1}{(-1+q)^2q^5(1+q)\lambda^7}\big[q(-1+\lambda)^2-(-1+\lambda)^2\lambda+q^9\lambda^4(-1+\lambda^2)+q^3(-1+\lambda)^2\lambda^2(1+\lambda+\lambda^2)+q^6\lambda^4(-1-\lambda+2\lambda^2)+q^5\lambda^2(-1+\lambda-\lambda^2+\lambda^3)-q^8\lambda^3(-1+\lambda-\lambda^2+\lambda^3)+q^4\lambda(-1+\lambda+\lambda^2-\lambda^5)+q^7(\lambda^3-\lambda^6)\big]$$

$$V_{\Box\Box}^{U(N)}[6_1] = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^2q^4(1+q)\lambda^5}\big[(-1+\lambda)\lambda+q(-1+\lambda)^2(1+\lambda)+q^2(-1+\lambda)^2\lambda(1+\lambda)+q^7\lambda^4(-3+\lambda^2)+2q^6\lambda^3(-1+\lambda^2)-2q^3\lambda^2(-1+\lambda+\lambda^2)+q^5\lambda^2(-1+2\lambda+4\lambda^2)-q^4\lambda(1-\lambda-3\lambda^2+2\lambda^3+\lambda^4)+q^8(\lambda^3-\lambda^5)\big]$$

$$V^{U(N)}_{\Box\Box}[6_2] = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^2q^6(1+q)\lambda^5}\Big[q+(-1+\lambda)\lambda-q\lambda^2-q^9(-4+\lambda)\lambda^2-q^{12}(-1+\lambda)\lambda^2-q^2(-1+\lambda)^2(1+\lambda)+q^3\lambda(-3+2\lambda)-2q^4(-1+\lambda^2)+q^{10}\lambda(-1+\lambda^2)+q^{11}\lambda^2(-1-\lambda+\lambda^2)+q^6(-1-3\lambda+4\lambda^2)+q^8\lambda(2-3\lambda^2+\lambda^3)+q^7(1-2\lambda-3\lambda^2+\lambda^4)+q^5(-1+3\lambda+2\lambda^2-2\lambda^3+\lambda^4)\Big]$$

$$V^{U(N)}_{\Box\Box}[6_3] = -\frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^2q^5(1+q)\lambda^3}\Big[-(-1+\lambda)\lambda+q^{12}(-1+\lambda)\lambda^2+q^3(1+3\lambda-4\lambda^2)+q(-1+\lambda^2)+q^9\lambda^2(-4+3\lambda+\lambda^2)+q^7(-1+4\lambda+\lambda^2-4\lambda^3)+q^5\lambda(-4+\lambda+4\lambda^2-\lambda^3)-q^4(2-3\lambda-3\lambda^2+\lambda^3)+q^{10}\lambda(1-\lambda-2\lambda^2+\lambda^3)+q^2(1-2\lambda-\lambda^2+\lambda^3)-q^8\lambda(1-3\lambda-3\lambda^2+2\lambda^3)+q^{11}(\lambda^2-\lambda^4)+q^6(1+\lambda-9\lambda^2+\lambda^3+\lambda^4)\Big]$$

$$V^{U(N)}_{\Box\Box}[7_2] = -\frac{(-1+\lambda)^2}{(-1+q)q^7\lambda^9}\Big[q-\lambda-q^2\lambda-2q^4\lambda+q^3(1+\lambda^2)-q^8\lambda(1+\lambda+2\lambda^2+2\lambda^3+\lambda^4)-q^6\lambda(2+\lambda+3\lambda^2+3\lambda^3+3\lambda^4+\lambda^5)+q^5(1+\lambda^2+\lambda^4+\lambda^5+\lambda^6)+q^7(1+\lambda+3\lambda^2+3\lambda^3+4\lambda^4+3\lambda^5+\lambda^6)\Big]$$

$$V^{U(N)}_{\Box\Box}[7_3] = \frac{(-1+\lambda)\lambda^3(-1+q\lambda)}{(-1+q)^2q^3(1+q)}\Big[1+q(-1+\lambda)-q^{12}(-1+\lambda)+\lambda-q^{14}(-1+\lambda)\lambda^2+q^{13}(-1+\lambda)^2\lambda(1+\lambda)-q^2(1+2\lambda)-q^4\lambda(-4+\lambda^2)-2q^3(-1+\lambda^2)+q^5(-2-\lambda+3\lambda^2)+q^8(-1+\lambda+4\lambda^2-2\lambda^3)+q^9(1-2\lambda+\lambda^3)+q^6(1-3\lambda-2\lambda^2+\lambda^3)+q^{10}\lambda(2-2\lambda-\lambda^2+\lambda^3)+q^7(1+3\lambda-3\lambda^2-2\lambda^3+\lambda^4)+q^{11}(-1+2\lambda^2-2\lambda^3+\lambda^4)\Big]$$

$$V^{U(N)}_{\Box\Box}[7_4] = \frac{(-1+\lambda)\lambda(-1+q\lambda)}{(-1+q)^2q(1+q)}\Big[-q^{10}(-1+\lambda)\lambda^4+(1+\lambda)^2+2q(-1+\lambda^2)+q^2(-1-6\lambda-\lambda^2+\lambda^3)+q^9\lambda^3(2-2\lambda-\lambda^2+\lambda^3)-q^3(-4-2\lambda+5\lambda^2+\lambda^3)+q^8\lambda^2(3-2\lambda-3\lambda^2+2\lambda^3)+q^5(-2-4\lambda+6\lambda^2+3\lambda^3-3\lambda^4)+q^7\lambda(2-2\lambda-4\lambda^2+3\lambda^3+\lambda^4)-q^4(1-6\lambda-4\lambda^2+4\lambda^3+\lambda^4)+q^6(1-2\lambda-4\lambda^2+5\lambda^3+\lambda^4-\lambda^5)\Big]$$

$$V^{U(N)}_{\Box\Box}[7_5] = == \frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^2q^9(1+q)\lambda^9}\Big[(-1+\lambda)\lambda-q^{13}(-1+\lambda)\lambda^3+q^2(-1+\lambda)^3(1+\lambda)+q^{14}\lambda^3(1+\lambda)-q^{12}\lambda^3(3+\lambda)+3q^{11}\lambda^2(-1+\lambda^2)+q^{10}\lambda(-1+6\lambda^2)-q^3\lambda(3-4\lambda+\lambda^3)+q(1-2\lambda^2+\lambda^3)+q^8\lambda(3-5\lambda-5\lambda^2+3\lambda^3)+q^9(\lambda+5\lambda^2-3\lambda^3-3\lambda^4)+q^4(2-\lambda-5\lambda^2+3\lambda^3+\lambda^4)+q^7(1-3\lambda-5\lambda^2+6\lambda^3+\lambda^4)-q^6(1+2\lambda-7\lambda^2+2\lambda^4)+q^5(-1+5\lambda-\lambda^2-5\lambda^3+2\lambda^4)\Big]$$

$$V^{U(N)}_{\square\square}[7_6] = \tfrac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^2q^6(1+q)\lambda^7}\big[(-1+\lambda)^2\lambda^2 - q^{12}(-1+\lambda)\lambda^4 + q^{11}\lambda^4(-2+\lambda^2) + q\lambda(-2+3\lambda+\lambda^2-2\lambda^3) - q^{10}\lambda^3(2-3\lambda^2+\lambda^3) - q^9\lambda^3(-1-7\lambda+2\lambda^2+\lambda^3) + q^7\lambda^2(3-7\lambda-7\lambda^2+4\lambda^3) + q^6\lambda^2(-4-6\lambda+10\lambda^2+\lambda^3-\lambda^4) + q^4\lambda(-1+8\lambda-\lambda^2-7\lambda^3+\lambda^4) + q^3\lambda(2-8\lambda^2+4\lambda^3+\lambda^4) + q^8\lambda^2(1+5\lambda-4\lambda^2-4\lambda^3+2\lambda^4) + q^5\lambda(-2-2\lambda+10\lambda^2+2\lambda^3-4\lambda^4+\lambda^5) - q^2(-1+\lambda+4\lambda^2-3\lambda^3-2\lambda^4+\lambda^5)\big]$$

$$V^{U(N)}_{\square\square}[7_7] = \tfrac{(-1+q\lambda)}{(-1+q)^2q^5(1+q)\lambda^3}\big[(-1+\lambda)^2\lambda + q^{12}(-1+\lambda)^3\lambda^2 - q(-1+\lambda)^2(1+2\lambda) - 2q^{11}(-1+\lambda)^2\lambda^2(-1-\lambda+\lambda^2) + q^3(-1+\lambda)^2(1+7\lambda+2\lambda^2) + q^5(-1+\lambda)^2(1-9\lambda-8\lambda^2+2\lambda^3) + q^9(-1+\lambda)^2\lambda(-1-8\lambda-\lambda^2+2\lambda^3) + q^2(2-5\lambda+\lambda^2+4\lambda^3-2\lambda^4) - q^7(-1+\lambda)^2(1-5\lambda-11\lambda^2+2\lambda^4) + q^8\lambda(-3+11\lambda-16\lambda^3+8\lambda^4) + q^6(2-18\lambda^2+15\lambda^3+5\lambda^4-4\lambda^5) + q^4(-4+6\lambda+8\lambda^2-12\lambda^3+\lambda^4+\lambda^5) + q^{10}\lambda(1-2\lambda-4\lambda^2+8\lambda^3-\lambda^4-3\lambda^5+\lambda^6)\big]$$

$$V^{U(N)}_{\square\square}[8_1] = \tfrac{1}{(-1+q)^2q^6(1+q)\lambda^7}\big[q(-1+\lambda)^2 - (-1+\lambda)^2\lambda - q^4(-1+\lambda)^2\lambda + q^3(-1+\lambda)^2\lambda^2 + q^5\lambda^5(-1+\lambda^3) + q^{11}\lambda^5(-1+\lambda+\lambda^3-\lambda^4) - q^6\lambda^4(-1+\lambda-\lambda^2+\lambda^4) + q^9\lambda^5(-1+\lambda-\lambda^2+\lambda^4) + q^8\lambda^4(-1-\lambda+2\lambda^4) + q^{10}\lambda^4(1-\lambda+2\lambda^2-\lambda^3-\lambda^4-\lambda^5+\lambda^6) + q^7(\lambda^4+\lambda^6-\lambda^8-\lambda^9)\big]$$

$$V^{U(N)}_{\square\square}[9_2] = \tfrac{1}{(-1+q)^2q^9(1+q)\lambda^{11}}\big[q(-1+\lambda)^2 - (-1+\lambda)^2\lambda - q^4(-1+\lambda)^2\lambda + q^3(-1+\lambda)^2\lambda^2 + q^{13}\lambda^6(-1+\lambda^4) + q^{10}\lambda^6(-1-\lambda+2\lambda^4) + q^9\lambda^4(-1+\lambda-\lambda^2+\lambda^5) - q^{12}\lambda^5(-1+\lambda-\lambda^2+\lambda^5) + q^7\lambda^5(1-\lambda-\lambda^4+\lambda^5) - q^8\lambda^4(1-2\lambda+\lambda^2-\lambda^3-\lambda^4+\lambda^5+\lambda^6) + q^{11}(\lambda^5+\lambda^7-\lambda^9-\lambda^{10})\big]$$

$$V^{U(N)}_{\square\square}[10_2] = \tfrac{1}{(-1+q)^2q^8(1+q)\lambda^9}\big[q(-1+\lambda)^2 - (-1+\lambda)^2\lambda - q^4(-1+\lambda)^2\lambda + q^3(-1+\lambda)^2\lambda^2 + q^7\lambda^6(-1+\lambda^4) + q^{13}\lambda^6(-1+\lambda+\lambda^4-\lambda^5) - q^8\lambda^5(-1+\lambda-\lambda^2+\lambda^5) + q^{11}\lambda^6(-1+\lambda-\lambda^2+\lambda^5) + q^{10}\lambda^5(-1-\lambda+2\lambda^5) - q^9\lambda^5(-1-\lambda^2+\lambda^5+\lambda^6) + q^{12}\lambda^5(1-\lambda+2\lambda^2-\lambda^3-\lambda^5-\lambda^6+\lambda^7)\big]$$

There seems to be a symmetry transformation on the polynomial variables which gives the $U(N)$ invariants of knots carrying antisymmetric second rank tensor representation $R = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. The symmetry relation for these non-torus knots (see also equation (7) in [15])

$$V^{U(N)}_{\square\square}[\mathcal{K}](q^{-1},\lambda) = V^{U(N)}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}[\mathcal{K}](q,\lambda). \tag{3.2}$$

Before we use these polynomial invariants in verifying Ooguri-Vafa conjecture, we shall enumerate the multi-colored link polynomials in the following section.

4. Link Polynomials

In this section we list the $U(N)$ link invariant for non torus links given in Figure 2 for representations $R_1, R_2 \in \{\square, \square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\}$.

1. For $R_1 = \square$, $R_2 = \square$:

$$V^{U(N)}_{(\square,\square)}[6_2] = \tfrac{1}{(-1+q)^2 q\lambda}\big[\lambda\left(-1+\lambda^2\right) + q^4\lambda\left(-1+\lambda^2\right) + q\left(1+\lambda-2\lambda^3\right) + q^3\left(1+\lambda-2\lambda^3\right) + q^2\left(-1-2\lambda+\lambda^2+2\lambda^3\right)\big]$$

$$V^{U(N)}_{(\square,\square)}[6_3] = \tfrac{(-1+\lambda)}{(-1+q)^2 q\lambda^3}\big[\lambda + q^4\lambda - q\left(1+3\lambda+2\lambda^2\right) - q^3\left(1+3\lambda+2\lambda^2\right) + q^2\left(2+4\lambda+3\lambda^2+\lambda^3\right)\big]$$

$$V^{U(N)}_{(\square,\square)}[7_1] = \tfrac{(-1+\lambda)}{(-1+q)^2 q^2\lambda^2}\big[-\lambda - q^6\lambda + q(1+\lambda)^2 + q^5(1+\lambda)^2 - q^2\left(2+3\lambda+\lambda^2\right) + q^3\left(2+3\lambda+\lambda^2\right) - q^4\left(2+3\lambda+\lambda^2\right)\big]$$

$$V^{U(N)}_{(\square,\square)}[7_2] = \tfrac{(-1+\lambda)}{(-1+q)^2 q^2\lambda^2}\big(-\lambda - q^6\lambda + q(1+\lambda)^2 - 2q^2(1+\lambda)^2 - 2q^4(1+\lambda)^2 + q^5(1+\lambda)^2 + q^3\left(2+5\lambda+3\lambda^2\right)\big]$$

$$V^{U(N)}_{(\square,\square)}[7_3] = \tfrac{(-1+\lambda)}{(-1+q)^2 q\lambda^3}\big[-\lambda(1+\lambda) - q^4\lambda(1+\lambda) + q(1+\lambda)^3 + q^3(1+\lambda)^3 - q^2\left(2+4\lambda+5\lambda^2+\lambda^3\right)\big]$$

2. For $R_1 = \square$, $R_2 = \square\square$:

$$V^{U(N)}_{(\square,\square\square)}[6_2] = \tfrac{(-1+\lambda)}{(-1+q)^3\sqrt{q}(1+q)\lambda^{3/2}}\big[q - \lambda - q^7\lambda^3 + q^8\lambda^3 + q^2\lambda(1+\lambda+\lambda^2) - q^6\lambda(1+\lambda+\lambda^2) + q^5\lambda(1+2\lambda+2\lambda^2) - q^4(-1+\lambda^3) - q^3(1+2\lambda+\lambda^2+\lambda^3)\big]$$

$$V^{U(N)}_{(\square,\square\square)}[6_3] = \tfrac{(-1+\lambda)}{(-1+q)^3 q^{5/2}(1+q)\lambda^{7/2}}\big[-\lambda + q^7\lambda^2 - q^2(1+\lambda) + q^5\lambda^3(1+\lambda) + q(1+\lambda)^2 - q^6\lambda(1+\lambda+2\lambda^2) + q^4(1+3\lambda+2\lambda^2+\lambda^3) - q^3(1+2\lambda+2\lambda^2+2\lambda^3)\big]$$

$$V^{U(N)}_{(\square,\square\square)}[7_1] = \tfrac{1}{(-1+q)^3 q^{5/2}(1+q)\lambda^{5/2}}\big[q + q^2(-1+\lambda) + (-1+\lambda)\lambda - q^{10}(-1+\lambda)\lambda^2 - q\lambda^3 - q^8(-1+\lambda)\lambda^3 + q^7(1+\lambda-2\lambda^3) - 2q^4(-1+\lambda^3) + q^9\lambda(-1+\lambda^3) + q^5(-1+\lambda+\lambda^3-\lambda^4) + q^3(-1-\lambda+\lambda^2+\lambda^4) + q^6(-1-\lambda+\lambda^3+\lambda^4)\big]$$

$$V^{U(N)}_{(\square,\square\square)}[7_2] = -\tfrac{(-1+q\lambda)}{(-1+q)^3 q^{7/2}(1+q)\lambda^{5/2}}\big[(q + (-1+\lambda)\lambda + q^9(-1+\lambda)\lambda - q\lambda^2 + q^5(-1+3\lambda+\lambda^2-3\lambda^3) + q^8(\lambda-\lambda^3) - q^2(1-2\lambda+\lambda^3) + q^4(2+\lambda-4\lambda^2+\lambda^3) + q^7(1-2\lambda^2+\lambda^3) + q^3(-1-3\lambda+3\lambda^2+\lambda^3) + q^6(-1-3\lambda+3\lambda^2+\lambda^3)\big]$$

$$V^{U(N)}_{(\Box,\Box\Box)}[7_3] = \frac{1}{(-1+q)^3q^{3/2}(1+q)\lambda^{7/2}}\Big[q^2(-1+\lambda^2)+\lambda(-1+\lambda^2)+q^4(1+2\lambda-3\lambda^4)+q^7(\lambda^2-\lambda^4)-q(-1-\lambda+\lambda^3+\lambda^4)+q^6\lambda(-1-\lambda+\lambda^3+\lambda^4)+q^5(\lambda^3-\lambda^5)+q^3(-1-\lambda-2\lambda^2+2\lambda^3+\lambda^4+\lambda^5)\Big]$$

Interchanging R_1, R_2 on the two components of the link, gives the same polynomial:

$$V^{U(N)}_{(\Box,\Box\Box)}[\mathcal{L}] = V^{U(N)}_{(\Box\Box,\Box)}[\mathcal{L}] \ , \tag{4.1}$$

and replacing the second rank symmetric representation $\Box\Box$ by antisymmetric representation $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$, the link polynomials are related as follows:

$$V^{U(N)}_{(\Box,\Box\Box)}[\mathcal{L}](q^{-1},\lambda) = -V^{U(N)}_{(\Box,\begin{smallmatrix}\Box\\\Box\end{smallmatrix})}[\mathcal{L}](q,\lambda) \tag{4.2}$$

3. For $R_1 = \Box\Box$, $R_2 = \Box\Box$:

$$V^{U(N)}_{(\Box\Box,\Box\Box)}[6_2] = \frac{1}{(-1+q)^4q(1+q)^2\lambda^2}\Big[-q^2(-1+\lambda)^2-(-1+\lambda)^2\lambda+q(-1+\lambda)^2(1+\lambda)+q^{15}\lambda^4(-1+\lambda^2)-q^{13}\lambda^4(-2+\lambda+\lambda^2)+q^3(-1+\lambda)^2\lambda(-2+\lambda+\lambda^2+\lambda^3)+q^{12}\lambda^3(-2-\lambda-2\lambda^2+5\lambda^3)+q^{11}\lambda^2(-1+\lambda-2\lambda^2+4\lambda^3-2\lambda^4)+q^{10}\lambda(-1+2\lambda+\lambda^2+3\lambda^3-5\lambda^5)+q^8\lambda(1-4\lambda+\lambda^2-3\lambda^3+4\lambda^4+\lambda^5)+q^5(-1+4\lambda-4\lambda^2-\lambda^4+2\lambda^5)+q^9\lambda(1+2\lambda-3\lambda^2-5\lambda^4+5\lambda^5)+q^7(1-4\lambda+\lambda^2+2\lambda^3+3\lambda^4+\lambda^5-4\lambda^6)+q^{14}(\lambda^3+\lambda^5-2\lambda^6)+2q^4(1-2\lambda+\lambda^3+\lambda^5-\lambda^6)+q^6(-1+\lambda+5\lambda^2-4\lambda^3+\lambda^4-5\lambda^5+3\lambda^6)\Big]$$

$$V^{U(N)}_{(\Box\Box,\Box\Box)}[6_3] = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^4q^7(1+q)^2\lambda^6}\Big[(-1+\lambda)\lambda+q^{12}(-1+\lambda)^2\lambda^2+q(1+\lambda-3\lambda^2)+q^2(-2+4\lambda+\lambda^2-2\lambda^3)+q^{11}\lambda^2(-3+2\lambda+3\lambda^2-2\lambda^3)+q^3(-1-5\lambda+9\lambda^2+3\lambda^3)+q^8\lambda(2-13\lambda-5\lambda^2+10\lambda^3)+q^6(-2+\lambda+19\lambda^2-2\lambda^3-6\lambda^4)+q^4(4-5\lambda-11\lambda^2+5\lambda^3+\lambda^4)+q^5(-1+9\lambda-8\lambda^2-12\lambda^3+2\lambda^4)+q^9\lambda(2+7\lambda-10\lambda^2-3\lambda^3+3\lambda^4)+q^{10}\lambda(-1+2\lambda+6\lambda^2-6\lambda^3-\lambda^4+\lambda^5)-q^7(-1+7\lambda+2\lambda^2-17\lambda^3+\lambda^4+2\lambda^5)\Big]$$

$$V^{U(N)}_{(\Box\Box,\Box\Box)}[7_1] = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^4q^6(1+q)^2\lambda^4}\Big[(-1+\lambda)\lambda-q^{18}(-1+\lambda)\lambda^2+q(1+\lambda-2\lambda^2)+q^{17}\lambda^2(-2+\lambda^2)+q^3\lambda(-4+3\lambda+\lambda^2)+q^2(-2+2\lambda+\lambda^2-\lambda^3)+q^7(5+5\lambda-7\lambda^2+\lambda^3)+q^4(3+2\lambda-5\lambda^2+\lambda^3)-q^{15}\lambda(-1-6\lambda+2\lambda^2+\lambda^3)+q^{14}\lambda(3-5\lambda-4\lambda^2+2\lambda^3)+q^{13}(1-5\lambda-7\lambda^2+5\lambda^3)+q^{12}(-2-3\lambda+11\lambda^2+2\lambda^3-2\lambda^4)+q^6(1-8\lambda+3\lambda^2+\lambda^3-\lambda^4)+q^{11}(-1+10\lambda+2\lambda^2-6\lambda^3+\lambda^4)-q^{16}(\lambda-3\lambda^3+\lambda^4)-q^9(2+11\lambda-5\lambda^2-3\lambda^3+\lambda^4)+q^5(-4+3\lambda+2\lambda^2-3\lambda^3+\lambda^4)+q^8(-5+7\lambda+6\lambda^2-3\lambda^3+\lambda^4)+q^{10}(5-\lambda-12\lambda^2+\lambda^3+\lambda^4)\Big]$$

$$V^{U(N)}_{(\square\!\square,\square\!\square)}[7_2] = \tfrac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^4q^8(1+q)^2\lambda^4}\big[((-1+\lambda)\lambda - q^{18}(-1+\lambda)\lambda^2 + q(1+\lambda-2\lambda^2) + q^{17}\lambda^2(-2+\lambda+\lambda^2) + q^{15}(\lambda+4\lambda^2-6\lambda^3) + q^{16}\lambda(-1+\lambda+2\lambda^2 - 2\lambda^3) + q^2(-2+3\lambda+\lambda^2-\lambda^3) + q^3(-1-6\lambda+5\lambda^2+\lambda^3) + q^4(5 - 2\lambda-9\lambda^2+2\lambda^3) + q^{14}\lambda(2-8\lambda+2\lambda^2+5\lambda^3) + q^{11}\lambda(6-19\lambda-4\lambda^2 + 9\lambda^3) + q^7(5-13\lambda-12\lambda^2+12\lambda^3) + q^9(-4+3\lambda+25\lambda^2-9\lambda^3 - 5\lambda^4) + q^{13}(1-5\lambda+3\lambda^2+10\lambda^3-5\lambda^4) + q^{12}(-2+3\lambda+12\lambda^2-14\lambda^3 - 3\lambda^4) + q^{10}(3-12\lambda+20\lambda^3-3\lambda^4) + q^6(-5-5\lambda+19\lambda^2+\lambda^3-2\lambda^4) + q^5(-2+13\lambda-2\lambda^2-6\lambda^3+\lambda^4) + q^8(1+13\lambda-17\lambda^2-12\lambda^3+5\lambda^4)\big]$$

$$V^{U(N)}_{(\square\!\square,\square\!\square)}[7_3] = \tfrac{(-1+\lambda)(-1+q\lambda)}{(-1+q)^4q^6(1+q)^2\lambda^6}\big[(-1+\lambda)\lambda - q^{12}\lambda^3(4-4\lambda-3\lambda^2+\lambda^3) + q(1+\lambda - 3\lambda^2+\lambda^3) + q^{13}\lambda^3(-1-3\lambda+\lambda^2+\lambda^3) + q^2(-2+4\lambda-3\lambda^3+\lambda^4) - q^{11}\lambda^2(2-8\lambda-8\lambda^2+5\lambda^3+\lambda^4) - q^3(1+5\lambda-9\lambda^2+3\lambda^4) + q^9\lambda(2+3\lambda - 20\lambda^2-4\lambda^3+7\lambda^4) + q^{14}(\lambda^3-\lambda^5) + q^4(4-5\lambda-8\lambda^2+11\lambda^3+\lambda^4-\lambda^5) - q^8\lambda(-2+13\lambda-3\lambda^2-22\lambda^3+\lambda^4+\lambda^5) + q^5(-1+9\lambda-10\lambda^2-10\lambda^3 + 9\lambda^4+\lambda^5) + q^6(-2+\lambda+16\lambda^2-13\lambda^3-12\lambda^4+2\lambda^5) + q^{10}\lambda(-1+4\lambda + 5\lambda^2-15\lambda^3-3\lambda^4+2\lambda^5) + q^7(1-7\lambda+3\lambda^2+22\lambda^3-7\lambda^4-5\lambda^5+\lambda^6)\big]$$

Changing both the rank two symmetric representation $\square\!\square$ by antisymmetric representation $\substack{\square\\\square}$, we find the following relation between the link polynomials:

$$V^{U(N)}_{(\square\!\square,\square\!\square)}[\mathcal{L}](q,\lambda) = V^{U(N)}_{(\substack{\square\\\square},\substack{\square\\\square})}[\mathcal{L}](q^{-1},\lambda). \tag{4.3}$$

With these polynomial invariants available for the non-torus knots and links in Figures 1, 2, we are in a position to verify Ooguri-Vafa[17] and Labastida-Marino-Vafa [18] conjectures.

5. Reformulated link invariants

In this section we explcitly write the reformulated link invariant for the non torus knots and links in Figure 1 and Figure 2. Rewriting the most general form of reformulated invariants $f_R[\mathcal{K}]$ and $f_{R_1,R_2}[\mathcal{L}]$ (see equation (3.16) in [23]) for representations $\square$ and $\square\!\square$ on the component knots, the expression for knots are

$$f_{\square}[\mathcal{K}] = V_{\square}[\mathcal{K}] \tag{5.1}$$

$$f_{\square\!\square}[\mathcal{K}] = V_{\square\!\square}[\mathcal{K}] - \frac{1}{2}\left(V_{\square}[\mathcal{K}]^2 + V^{(2)}_{\square}[\mathcal{K}]\right) \tag{5.2}$$

$$f_{\substack{\square\\\square}}[\mathcal{K}] = V_{\substack{\square\\\square}}[\mathcal{K}] - \frac{1}{2}\left(V_{\square}[\mathcal{K}]^2 - V^{(2)}_{\square}[\mathcal{K}]\right), \tag{5.3}$$

where we have suppressed $U(N)$ superscript on the knot invariants ($V_R[\mathcal{K}] \equiv V^{U(N)}_R[\mathcal{K}](q,\lambda)$). Further, we use the notation $V^{(n)}_R[\mathcal{K}] \equiv V_R[\mathcal{K}](q^n,\lambda^n)$. Ooguri-Vafa conjecture [17] states that the reformulated invariants for knots should have the following structure

$$f_R(q,\lambda) = \sum_{s,Q} \frac{N_{Q,R,s}}{q^{1/2}-q^{-1/2}} a^Q q^s, \tag{5.4}$$

where $N_{Q,R,s}$ are integer and Q and s are, in general, half integers. Clearly for $R = \Box$ (fundamental representation), the polynomial structure in section 3 for $V_{\Box}^{U(N)}[\mathcal{K}]$ (5.1) obeys eqn.(5.4). We will verify for $R = \Box\Box$ and $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ in the following subsection.

The reformulated invariants in terms of two-component link invariants has the following form for $R_1, R_2 \in \{\Box, \Box\Box, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}\}$:

$$\begin{aligned}
f_{(\Box,\Box)}[\mathcal{L}] &= V_{(\Box,\Box)}[\mathcal{L}] - V_{\Box}[\mathcal{K}_1]V_{\Box}[\mathcal{K}_2] && (5.5)\\
f_{(\Box,\Box\Box)}[\mathcal{L}] &= V_{(\Box,\Box\Box)}[\mathcal{L}] - V_{(\Box,\Box)}[\mathcal{L}]V_{\Box}[\mathcal{K}_2] - V_{\Box}[\mathcal{K}_1]V_{\Box\Box}[\mathcal{K}_2]\\
&\quad + V_{\Box}[\mathcal{K}_1]V_{\Box}[\mathcal{K}_2]^2 && (5.6)\\
f_{(\Box,\begin{smallmatrix}\Box\\\Box\end{smallmatrix})}[\mathcal{L}] &= V_{(\Box,\begin{smallmatrix}\Box\\\Box\end{smallmatrix})}[\mathcal{L}] - V_{(\Box,\Box)}[\mathcal{L}]V_{\Box}[\mathcal{K}_2] - V_{\Box}[\mathcal{K}_1]V_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}[\mathcal{K}_2]\\
&\quad + V_{\Box}[\mathcal{K}_1]V_{\Box}[\mathcal{K}_2]^2 && (5.7)\\
f_{(\Box\Box,\Box)}[\mathcal{L}] &= V_{(\Box\Box,\Box)}[\mathcal{L}] - V_{(\Box,\Box)}[\mathcal{L}]V_{\Box}[\mathcal{K}_1] - V_{\Box\Box}[\mathcal{K}_1]V_{\Box}[\mathcal{K}_2]\\
&\quad + V_{\Box}[\mathcal{K}_1]^2V_{\Box}[\mathcal{K}_2] && (5.8)\\
f_{(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},\Box)}[\mathcal{L}] &= V_{(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},\Box)}[\mathcal{L}] - V_{(\Box,\Box)}[\mathcal{L}]V_{\Box}[\mathcal{K}_1] - V_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}[\mathcal{K}_1]V_{\Box}[\mathcal{K}_2]\\
&\quad + V_{\Box}[\mathcal{K}_1]^2V_{\Box}[\mathcal{K}_2] && (5.9)
\end{aligned}$$

$$\begin{aligned}
f_{(\Box\Box,\Box\Box)}[\mathcal{L}] &= V_{(\Box\Box,\Box\Box)}[\mathcal{L}] - V_{\Box\Box}(\mathcal{K}_1)V_{\Box\Box}[\mathcal{K}_2] - V_{(\Box\Box,\Box)}[\mathcal{L}]V_{\Box}[\mathcal{K}_2]\\
&\quad - V_{(\Box,\Box\Box)}[\mathcal{L}]V_{\Box}[\mathcal{K}_1] - \frac{1}{2}V_{(\Box,\Box)}[\mathcal{L}]^2 + 2V_{(\Box,\Box)}[\mathcal{L}]V_{\Box}[\mathcal{K}_1]V_{\Box}[\mathcal{K}_2]\\
&\quad + V_{\Box}[\mathcal{K}_1]^2V_{\Box\Box}[\mathcal{K}_2] + V_{\Box\Box}[\mathcal{K}_1]V_{\Box}[\mathcal{K}_2]^2 - \frac{3}{2}V_{(\Box)}[\mathcal{K}_1]^2V_{(\Box)}[\mathcal{K}_2]^2\\
&\quad - \frac{1}{2}V^{(2)}_{(\Box,\Box)}[\mathcal{L}] + \frac{1}{2}V^{(2)}_{\Box}[\mathcal{K}_1]V^{(2)}_{\Box}[\mathcal{K}_2] && (5.10)
\end{aligned}$$

$$\begin{aligned}
f_{(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},\begin{smallmatrix}\Box\\\Box\end{smallmatrix})}[\mathcal{L}] &= V_{(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},\begin{smallmatrix}\Box\\\Box\end{smallmatrix})}[\mathcal{L}] - V_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}[\mathcal{K}_1]V_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}[\mathcal{K}_2] - V_{(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},\Box)}[\mathcal{L}]V_{\Box}[\mathcal{K}_2]\\
&\quad - V_{(\Box,\begin{smallmatrix}\Box\\\Box\end{smallmatrix})}[\mathcal{L}]V_{\Box}[\mathcal{K}_1] - \frac{1}{2}V_{(\Box,\Box)}[\mathcal{L}]^2 + 2V_{(\Box,\Box)}[\mathcal{L}]V_{\Box}[\mathcal{K}_1]V_{\Box}[\mathcal{K}_2]\\
&\quad + V_{\Box}[\mathcal{K}_1]^2V_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}[\mathcal{K}_2] + V_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}[\mathcal{K}_1]V_{\Box}[\mathcal{K}_2]^2 - \frac{3}{2}V_{(\Box)}[\mathcal{K}_1]^2V_{(\Box)}[\mathcal{K}_2]^2\\
&\quad - \frac{1}{2}V^{(2)}_{(\Box,\Box)}[\mathcal{L}] + \frac{1}{2}V^{(2)}_{\Box}[\mathcal{K}_1]V^{(2)}_{\Box}[\mathcal{K}_2] && (5.11)
\end{aligned}$$

Here the components knots $\mathcal{K}_1$ and $\mathcal{K}_2$ are unknots for the non-torus links in Figure 2. The generalisation of Ooguri-Vafa conjecture for links was proposed in [18] which states that reformulated invariants for r-component link should have the following structure

$$f_{(R_1,R_2,\ldots R_r)}(q,\lambda) = (q^{1/2}-q^{-1/2})^{r-2} \sum_{Q,s} N_{(R_1,\ldots R_r),Q,s} q^s \lambda^Q \ , \tag{5.12}$$

where $N_{(R_1,\cdots R_r),Q,s}$ are integer and Q and s are half integers. We can see below that all the reformulated invariants we calculate indeed satisfy the conjecture.

5.1 Reformulated invariant for knots

We have already seen in section 3, $V_{\Box}[\mathcal{K}]$ has the Ooguri-Vafa form given in eqn.(5.4). For the symmetric second rank tensor $R = \Box\Box$ placed on the knot, $f_{\Box\Box}[\mathcal{K}]$ are:

$$f_{\Box\Box}[4_1] = \tfrac{(-1+\lambda)^2}{(-1+q)q^2\lambda^3}\left[-q+\lambda-q^5\lambda^3+q^4\lambda^4+q^3\lambda(1+\lambda)-q^2\lambda^2(1+\lambda)\right]$$

$$f_{\Box\Box}[5_2] = -\tfrac{(1-q+q^2)(-1+\lambda)^2}{(-1+q)q^5\lambda^7}\left[q(-1+\lambda)+q^2(-1+\lambda)+\lambda+q^4\lambda(1+\lambda+\lambda^2)-q^3(1+\lambda^2+\lambda^3+\lambda^4)\right]$$

$$f_{\Box\Box}[6_1] = -\tfrac{(-1+\lambda)^2}{(-1+q)q^4\lambda^5}\left[q-\lambda-q^2\lambda+q^7\lambda^4(1+\lambda)+q^3(1+\lambda^2)-q^5\lambda(1+\lambda+\lambda^2)^2+q^4\lambda(-1+\lambda+2\lambda^2+2\lambda^3+\lambda^4)+q^6(\lambda^2+\lambda^3+\lambda^4-\lambda^5-\lambda^6\right]$$

$$f_{\Box\Box}[6_2] = -\tfrac{(1-q+q^2)(-1+\lambda)^2}{(-1+q)q^6\lambda^5}\left[-\lambda-q^5\lambda+q^2(-1+\lambda)\lambda+q^9\lambda^3-q^8(-1+\lambda)\lambda^3+q^6\lambda^2(1+\lambda)+q(1+\lambda^2)-q^3\lambda(2+\lambda^2)+q^4(1+2\lambda^2)-q^7\lambda(1+\lambda+\lambda^3)\right]$$

$$f_{\Box\Box}[6_3] = -\tfrac{(1-q+q^2)(-1+\lambda)^2}{(-1+q)q^5\lambda^3}\left[-\lambda+q^2\lambda^2-q^7\lambda^2+q^9\lambda^3+q(1+\lambda^2)+q^3(-1-\lambda+\lambda^2)+q^6\lambda^2(-1+\lambda+\lambda^2)-q^4\lambda(2+2\lambda+2\lambda^2+\lambda^3)+q^5(1+2\lambda+2\lambda^2+2\lambda^3)-q^8(\lambda^2+\lambda^4)\right]$$

$$f_{\Box\Box}[7_2] = -\tfrac{(-1+\lambda)^2}{(-1+q)q^7\lambda^9}\left[q-\lambda-q^2\lambda-2q^4\lambda+q^3(1+\lambda^2)-q^8\lambda(1+\lambda+2\lambda^2+2\lambda^3+\lambda^4)-q^6\lambda(2+\lambda+3\lambda^2+3\lambda^3+3\lambda^4+\lambda^5)+q^5(1+\lambda^2+\lambda^4+\lambda^5+\lambda^6)+q^7(1+\lambda+3\lambda^2+3\lambda^3+4\lambda^4+3\lambda^5+\lambda^6)\right]$$

$$f_{\Box\Box}[7_3] = \tfrac{(-1+\lambda)^2\lambda^3}{(-1+q)q^2}\left[-2q^{10}\lambda^3-q^{12}\lambda^3+q^{11}\lambda^4-\lambda(1+\lambda+\lambda^2)+q^6\lambda(-1-\lambda-5\lambda^2+\lambda^3)+q^4\lambda(-1-\lambda-4\lambda^2+\lambda^3)-2q^2(1+2\lambda+2\lambda^2+2\lambda^3)+q^5\lambda(1+2\lambda-\lambda^2+3\lambda^3)+q^8(-1-\lambda-\lambda^2-3\lambda^3+\lambda^4)+q(1+3\lambda+3\lambda^2+\lambda^3+\lambda^4)+q^3(2+3\lambda+4\lambda^2+2\lambda^4)+q^9(1+\lambda+\lambda^2-\lambda^3+2\lambda^4)+q^7(1+2\lambda+2\lambda^2-\lambda^3+2\lambda^4)\right]$$

$$f_{\Box\Box}[7_4] = \tfrac{(-1+\lambda)^2\lambda}{(-1+q)q}\left[-q^9\lambda^5+\lambda^2(1+\lambda)-q^7\lambda^3(1+\lambda)+q^8\lambda^4(-1+\lambda^2)+q^6\lambda^2(1+\lambda+\lambda^2+\lambda^3+\lambda^4)-q\lambda(2+4\lambda+4\lambda^2+3\lambda^3+\lambda^4)+q^5(\lambda+\lambda^2-2\lambda^4-\lambda^5)-q^3(2+5\lambda+5\lambda^2+5\lambda^3+4\lambda^4+\lambda^5)+q^4(1+\lambda+2\lambda^2+\lambda^3-\lambda^4+\lambda^5+\lambda^6)+q^2(1+5\lambda+8\lambda^2+7\lambda^3+3\lambda^4+2\lambda^5+\lambda^6)\right]$$

$$f_{\Box\Box}[7_5] = \tfrac{(1-q+q^2)(-1+\lambda)^2}{(-1+q)q^9\lambda^9}\left[-q+\lambda-q^7(-1+\lambda)\lambda^3+q^8\lambda(1+\lambda)^3+q^{10}\lambda(1+2\lambda+2\lambda^2)+q^4(-1+\lambda+\lambda^2+2\lambda^3)+q^6(\lambda^3-2\lambda^4)+q^3(\lambda-\lambda^4)-q^5(1+\lambda+\lambda^2+\lambda^4)-q^9(1+2\lambda+3\lambda^2+3\lambda^3+2\lambda^4)\right]$$

$$f_{\square\square}[7_6] = -\frac{(-1+\lambda)^2}{(-1+q)q^6\lambda^7}\big[-(-1+\lambda)\lambda^2+q^{11}\lambda^5-q^{10}\lambda^5(1+\lambda)+q^9\lambda^3(-2-2\lambda-\lambda^2+\lambda^3)+q\lambda(-2+\lambda+\lambda^2+\lambda^3)-q^3\lambda(1-3\lambda+\lambda^2+\lambda^4)-q^2(-1+\lambda^2+3\lambda^3+\lambda^4)+q^8\lambda^2(2+4\lambda+5\lambda^2+4\lambda^3+\lambda^4)-q^5\lambda(3+4\lambda+8\lambda^2+6\lambda^3+2\lambda^4)-q^7\lambda(2+2\lambda+7\lambda^2+6\lambda^3+4\lambda^4+2\lambda^5)+q^4(1-\lambda+4\lambda^2+3\lambda^3+5\lambda^4+2\lambda^5)+q^6(1+\lambda+6\lambda^2+6\lambda^3+7\lambda^4+2\lambda^5+\lambda^6)\big]$$

$$f_{\square\square}[7_7] = \frac{(-1+\lambda)^2}{(-1+q)q^5\lambda^3}\big[\lambda+2q^3\lambda+q^{11}(-1+\lambda)\lambda^3-q(1+\lambda)^2+q^2(2+\lambda)-q^8\lambda^2(-1+\lambda-2\lambda^2+\lambda^3)+q^9\lambda^2(-1-2\lambda-\lambda^2-\lambda^3+\lambda^4)-q^5\lambda(7+12\lambda+12\lambda^2+5\lambda^3+2\lambda^4)+q^4(-2+5\lambda^2+5\lambda^3+4\lambda^4)+q^{10}(\lambda^2+\lambda^3+2\lambda^4-2\lambda^5)+q^6(2+8\lambda+11\lambda^2+9\lambda^3+4\lambda^4-\lambda^5)+q^7(-1-3\lambda-4\lambda^2-3\lambda^3+3\lambda^4-\lambda^5+\lambda^6)\big]$$

$$f_{\square\square}[8_1] = -\frac{(-1+\lambda)^2}{(-1+q)q^6\lambda^7}\big[q-\lambda-q^2\lambda-2q^4\lambda+q^3(1+\lambda^2)+q^5(1+\lambda^2)+q^9\lambda^5(1+\lambda+\lambda^2)-q^8\lambda^2(-1-2\lambda-2\lambda^2-2\lambda^3+\lambda^5+\lambda^6)+q^6\lambda(-1+\lambda+2\lambda^2+3\lambda^3+3\lambda^4+2\lambda^5+\lambda^6)-q^7\lambda(1+2\lambda+4\lambda^2+5\lambda^3+4\lambda^4+3\lambda^5+\lambda^6)\big]$$

$$f_{\square\square}[9_2] = -\frac{(-1+\lambda)^2}{(-1+q)q^9\lambda^{11}}\big[q-\lambda-q^2\lambda-2q^4\lambda-2q^6\lambda+q^3(1+\lambda^2)+q^5(1+\lambda^2)-q^{10}\lambda(1+\lambda+2\lambda^2+3\lambda^3+3\lambda^4+2\lambda^5+\lambda^6)-q^8\lambda(2+\lambda+2\lambda^2+4\lambda^3+4\lambda^4+4\lambda^5+3\lambda^6+\lambda^7)+q^7(1+\lambda^2+\lambda^5+\lambda^6+\lambda^7+\lambda^8)+q^9(1+\lambda+3\lambda^2+4\lambda^3+5\lambda^4+6\lambda^5+5\lambda^6+3\lambda^7+\lambda^8)\big]$$

$$f_{\square\square}[10_1] = -\frac{(-1+\lambda)^2}{(-1+q)q^8\lambda^9}\big[q-\lambda-q^2\lambda-2q^4\lambda-2q^6\lambda+q^3(1+\lambda^2)+q^5(1+\lambda^2)+q^7(1+\lambda^2)+q^{11}\lambda^6(1+\lambda+\lambda^2+\lambda^3)+q^{10}\lambda^2(1+2\lambda+3\lambda^2+3\lambda^3+3\lambda^4+\lambda^5-\lambda^7-\lambda^8)+q^8\lambda(-1+\lambda+2\lambda^2+3\lambda^3+4\lambda^4+4\lambda^5+3\lambda^6+2\lambda^7+\lambda^8)-q^9\lambda(1+2\lambda+4\lambda^2+6\lambda^3+7\lambda^4+6\lambda^5+5\lambda^6+3\lambda^7+\lambda^8\big]$$

Changing the symmetric representation by antisymmetry representation, we find the following relation between the reformulated invariants:

$$f_{\square\square}[\mathcal{K}](q^{-1},\lambda) = f_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}[\mathcal{K}](q,\lambda). \tag{5.13}$$

5.2 Reformulated invariant for links

1. For $R_1 = \square$, $R_2 = \square$:

$$f_{(\square,\square)}[6_2] = \frac{(-1+\lambda)}{q\lambda}\big[-q+\lambda+q^2\lambda+\lambda^2+q^2\lambda^2\big]$$

$$f_{(\square,\square)}[6_3] = \frac{(-1+\lambda)}{q\lambda^3}\left[\lambda+q^2\lambda-q\left(1+\lambda+2\lambda^2\right)\right]$$

$$f_{(\square,\square)}[7_1] = \frac{(-1+\lambda)}{q^2\lambda^2}\left[-\lambda-q^4\lambda+q^2(-2+\lambda)\lambda+q(1+\lambda^2)+q^3(1+\lambda^2)\right]$$

$$f_{(\square,\square)}[7_2] = \frac{(-1+\lambda)}{q^2\lambda^2}\left[(-\lambda-3q^2\lambda-q^4\lambda+q(1+\lambda^2)+q^3(1+\lambda^2)\right]$$

$$f_{(\square,\square)}[7_3] = \frac{\left(-1+\lambda^2\right)}{q\lambda^3}\left[q-\lambda-q^2\lambda+q\lambda^2\right]$$

2. For $R_1=\square$, $R_2=\square\square$:

$$f_{(\square,\square\square)}[6_2] = \frac{(-1+\lambda)}{\sqrt{q}\sqrt{\lambda}}\left[\lambda^2+q\lambda^2+q^3\lambda^2+q^4\lambda^2+q^2\left(-1-\lambda+\lambda^2\right)\right]$$

$$f_{(\square,\square\square)}[6_3] = \frac{1}{q^{5/2}\lambda^{7/2}}(1+q)(-1+\lambda)\left[\lambda+q^2\lambda-q\left(1+\lambda+\lambda^2\right)\right]$$

$$f_{(\square,\square\square)}[7_1] = \frac{(-1+\lambda)}{q^{5/2}\lambda^{5/2}}\left[q-\lambda-q^6\lambda^2+q^3(-2+\lambda)\lambda^2+q^2\lambda^3+q^4\lambda(1-\lambda+\lambda^2)+\right.$$
$$\left.q^5\lambda(1-\lambda+\lambda^2)\right]$$

$$f_{(\square,\square\square)}[7_2] = \frac{(-1+\lambda)}{q^{7/2}\lambda^{5/2}}\left[(q^4-\lambda-3q^3\lambda-q^6\lambda^2+q^5\lambda^3+q(1-\lambda+\lambda^2)+q^2(1-\lambda+\lambda^2)\right]$$

$$f_{(\square,\square\square)}[7_3] = \frac{\left(-1+\lambda^2\right)}{q^{3/2}\lambda^{7/2}}\left[q-\lambda-q^3\lambda^2+q^2\lambda^3\right]$$

We have checked that $f_{(\square,\square\square)}[L]=f_{(\square\square,\square)}[L]$ for these links. We also have the symmetry relation

$$f_{(\square,\square\square)}[\mathcal{L}](q^{-1},\lambda) = -f_{(\square,\begin{smallmatrix}\square\\\square\end{smallmatrix})}[\mathcal{L}](q,\lambda). \tag{5.14}$$

3. For $R_1=\square\square$, $R_2=\square\square$:

$$f_{(\square\square,\square\square)}[6_2] = \frac{1}{q^2\lambda}\left[q(-1+\lambda)^2\lambda+q^9(-1+\lambda)^2\lambda^2+\lambda^3-\lambda^5+q^7(-1+\lambda)^2\lambda^2(3+\right.$$
$$\lambda)+q^{10}\lambda^3(-1+\lambda^2)+q^8\lambda^2(2-3\lambda+\lambda^2)+q^5(-1+\lambda)^2(-1-2\lambda+$$
$$2\lambda^2)+q^3(-1+\lambda)^2(1+\lambda+2\lambda^2)+q^4(-1+\lambda+4\lambda^2-5\lambda^3+\lambda^4)+$$
$$\left.q^6\lambda(-1+5\lambda-8\lambda^2+3\lambda^3+\lambda^4)+q^2(-1+\lambda^2+\lambda^4-\lambda^5)\right]$$

$$f_{(\square\square,\square\square)}[6_3] = \frac{1}{q^7\lambda^6}\left[-(-1+\lambda)^2\lambda+q^2(-1+\lambda)^2\lambda^2-q^9(-1+\lambda)\lambda^3+q^5(-1+\lambda)^2\lambda^2(2+\right.$$
$$\lambda^2)-q^4(-1+\lambda)^2\lambda(1-\lambda+2\lambda^2)+q^8(-1+\lambda)^2\lambda(1+\lambda+2\lambda^2)+$$
$$q^3\lambda^2(3-4\lambda+3\lambda^2-2\lambda^3)+q^6(-1+\lambda)^2\lambda(1+3\lambda+2\lambda^2+2\lambda^3)+$$
$$\left.q(1-2\lambda+\lambda^2-\lambda^3+\lambda^4)-q^7(1+2\lambda^2-2\lambda^3-2\lambda^4+\lambda^6)\right]$$

$$
\begin{aligned}
f_{(\yng(2),\yng(2))}[7_1] = \;& \tfrac{1}{q^6\lambda^4}\big[-(-1+\lambda)^2\lambda + q^{11}(-1+\lambda)^4\lambda^2 - q^{13}(-1+\lambda)^2\lambda^3 + q(-1+\\
&\lambda)^2(1+\lambda^2)+3q^9(-1+\lambda)^2\lambda^2(1-\lambda+\lambda^2)+q^3(-1+\lambda)^2(1-\lambda+2\lambda^2)+\\
&q^2\lambda(-2+4\lambda-3\lambda^2+\lambda^3)+q^{12}\lambda^2(1-2\lambda+3\lambda^2-3\lambda^3+\lambda^4)-q^6(1+\\
&\lambda-\lambda^2-2\lambda^3+\lambda^4)+q^7(-1+\lambda)^2(1+2\lambda+2\lambda^2-\lambda^3+\lambda^4)+q^8\lambda^2(2-\\
&4\lambda+7\lambda^2-7\lambda^3+2\lambda^4)-q^5\lambda(-1+\lambda+2\lambda^3-3\lambda^4+\lambda^5)+q^{10}\lambda(-1+\\
&2\lambda-7\lambda^2+11\lambda^3-7\lambda^4+2\lambda^5)-q^4(-1+3\lambda-4\lambda^2+\lambda^3+\lambda^4-\lambda^5+\lambda^6)\big]
\end{aligned}
$$

$$
\begin{aligned}
f_{(\yng(2),\yng(2))}[7_2] = \;& -\tfrac{1}{q^8\lambda^4}\big[(-1+\lambda)^2\lambda+q^{13}(-1+\lambda)^2\lambda^3-q(-1+\lambda)^2(1+\lambda^2)+q^3(-1+\\
&\lambda)^2\lambda(3-2\lambda+\lambda^2)+q^{11}(-1+\lambda)^2\lambda(-1-\lambda+\lambda^2)+q^5(-1+\lambda)^2(-1+\\
&2\lambda-5\lambda^2+\lambda^3)-q^2\lambda(-1+4\lambda-4\lambda^2+\lambda^3)-q^{12}\lambda^3(-1+2\lambda-2\lambda^2+\\
&\lambda^3)-q^7(-1+\lambda)^2(1+\lambda+5\lambda^2+\lambda^3)+q^{10}\lambda^2(1-2\lambda-2\lambda^2+3\lambda^3)+\\
&q^6\lambda(3-10\lambda+9\lambda^2-5\lambda^3+3\lambda^4)+q^4(-1+6\lambda-11\lambda^2+12\lambda^3-7\lambda^4+\\
&\lambda^5)-q^9\lambda(3-3\lambda-2\lambda^3+\lambda^4+\lambda^5)+q^8(1+\lambda+2\lambda^2-5\lambda^3+\lambda^4-\lambda^5+\lambda^6)\big]
\end{aligned}
$$

$$
\begin{aligned}
f_{(\yng(2),\yng(2))}[7_3] = \;& -\tfrac{(-1+\lambda)^2}{q^6\lambda^6}\big[(-q+\lambda-q^5\lambda+q^9\lambda^4(1+\lambda)+q^4\lambda(1+\lambda)^2+q^6\lambda^2(1+\\
&\lambda)^2(1+\lambda^2)-q^3\lambda^2(1+\lambda+\lambda^2)-q^7\lambda(1+2\lambda+3\lambda^2+3\lambda^3+3\lambda^4+\\
&\lambda^5)+q^8(\lambda^2+\lambda^3+2\lambda^4-\lambda^6)\big]
\end{aligned}
$$

Changing both the rank two symmetric representation $\yng(2)$ by antisymmetric representation $\yng(1,1)$, we find the following relation between the reformulated invariants:

$$f_{(\yng(2),\yng(2))}[\mathcal{L}](q,\lambda) = f_{(\yng(1,1),\yng(1,1))}[\mathcal{L}](q^{-1},\lambda). \tag{5.15}$$

6. Conclusion

This paper presents the $U(N)$ Chern-Simons invariants for the non-torus knots and non-torus links in Figure 1 and Figure 2. We have written the explicit polynomial form for few representations ($\yng(1)$, $\yng(2)$, $\yng(1,1)$) and also obtained the reformulated invariants. For completeness, we have included reformulated invariants of knots $4_1, 6_1$ [5], knot 5_2 and link 6_1 [16]. The form of these reformulated invariants are consistent with the Ooguri-Vafa (5.4) and Labastida-Marino-Vafa conjectures (5.12).

Recently, the polynomial form for figure eight knot carrying totally symmetric representation is given in Ref.[15]. Further, the polynomial for the knot carrying totally antisymmetric representation can be obtained by suitable change of polynomial variable which we also observe when we change $R = \yng(2)$ to $R = \yng(1,1)$(3.2). Motivated by the conjecture of the polynomial form for figure eight, we have been trying to propose the polynomial forms for non-torus knots carrying nth-rank symmetric tensor representation. Particularly looking at the pattern of the polynomials, for a class of twist knots $\mathcal{K}_p$ carrying representations $R = \yng(1), \yng(2)$ and $\yng(1,1)$, we attempted to write $V_n^{U(N)}[\mathcal{K}_p]$ [24] where subscript n denotes the n-rank symmetric representation. We leave the reader to see the forthcoming paper [24, 25] for interesting results on twist knots.

There are other non-torus knots besides the twist knots in Figure 1. We hope that the results in Ref.[24], will suggest a closed form expression for $SU(N)$ quantum Racah coefficients similar to $SU(2)$ quantum Racah coefficients [26]. In fact, the closed form expression for $SU(N)$ quantum Racah coefficients will determine the colored polynomial invariant for any knot or any link carrying arbitrary representation R.

In the light of the recent developments on superpolynomials [15, 27, 24], which has an additional polynomial variable t, it should be a systematic exercise to look at the t-deformation of the quantum Racah coefficients. We hope to report on these aspects in future.

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